Lie algebras on hyperelliptic curves and finite-dimensional integrable systems

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Abstract

We construct a new family of infinite-dimensional quasi-graded Lie algebras on hyperelliptic curves. We show that constructed algebras possess infinite number of invariant functions and admit a decomposition into the direct sum of two subalgebras. These two facts together enables one to use them to construct new integrable finite-dimensional hamiltonian systems. In such a way we find new integrable hamiltonian systems, which are direct higher rank generalizations of the integrable systems of Steklov-Liapunov, associated with the e(3) algebra and Steklov-Veselov associated with the so(4) algebra.

1 Introduction

The main purpose of the present paper is to introduce new integrable systems of Euler-Arnold type on the finite-dimensional Lie algebras. It is known, that such systems admit Lax-pair representations:

$$\frac{dL(t)}{dt} = [L(t), M(t)], \tag{1}$$

where L and M are some matrix depending on the dynamical variables [1].

Almost in all cases Lax operator depend also on additional parameter — so called "spectral parameter" w: $L = L(w)$. This dependence permits to construct large number of integrals of motion via the expansion of functions $I^k(w) = TrL(w)^k$ in the series of the powers of the spectral parameter. Usually dependence of Lax operator on w is rational or elliptic.

The natural approach to construction of new types of integrable systems is search for the solutions of Lax equations with other (more complicated) dependence on the spectral parameter. We will solve this problem for the case of $L(w)$ with the hyperelliptic dependence on the spectral parameter.

Our approach is based on the usage of infinite-dimensional Lie algebras. It is known [4],[5],[6] that group theoretical explanation of the integrability of Lax equations on finite-dimensional Lie algebras with the rational spectral parameters is based on the loop algebras, i.e. algebras of the form $\mathfrak{g} \otimes P(\lambda, \lambda^{-1})$. In the works [7],[8], [9] it was shown, that Lax equation with the elliptic spectral parameters on the algebra $so(3)$ and some its extensions could be obtained from the infinite-dimensional Lie algebras of the special elliptic matrix-valued functions with the values in $so(3)$ via Kostant-Adler scheme [2]- [3].

We generalize construction of [8] on the case of classical matrix algebras of higher rank and investigate corresponding finite-dimensional integrable systems. Growth of the rank of algebra requires automatic growth of the genus of the curve. As a result we obtain algebras of $gl(n)$- , $so(n)$- and $sp(n)$- valued functions on the hyperelliptic curves of genus g, where $n = 2g + 2$ or $n = 2g + 1$, or, to be more precise, on its double covering. Obtained algebras have many nice properties. They are quasi-graded. They possess central extensions. They possess infinite number of invariant functions. In the rational degeneration of the curve they coincides with the ordinary loop algebras. So they could be viewed as "hyperelliptic " generalizations of the loop algebras.

There exists other higher genus generalization of the loop algebras — so called Krichiver-Novikov algebras [11] of the matrix-valued holomorphic functions on the Riemanian surfaces with two punctured points. But Krichiver-Novikov algebras do not admit Kostant-Adler scheme [12]. Hence this generalization can not be used for producing new integrable systems with the spectral parameter on the higher genus curves.

Contrary to the Krichiver-Novikov algebras, our algebras admit Kostant-Adler scheme, and hence, could be used for constructing integrable systems.

Using our algebra as hidden symmetry algebra we obtain new integrable systems on $\mathfrak{g} \oplus \mathfrak{g}$ that describes two interacting generalized rigid bodies. This system is direct higher rank generalization of integrable case of Steklov and Veselov [13] on $so(4) = so(3) \oplus so(3)$. We also obtain new integrable systems on the semidirect sum $\mathfrak{g} + \mathfrak{g}$, that generalize integrable case of Steklov-Ljapunov on $e(3)$ [14].

Structure of the algebra provides a possibility to give two- dimensional generalization of constructed integrable systems via finite-gap extension method [7]-[9]. We will return to this problem in our subsequent publications.

2 Quasi-graded algebras on hyperelliptic curves

2.1 Construction

Hyperelliptic curve embedded in $\mathbb{C}^n$. Let us consider in the space $\mathbb{C}^n$ with the coordinates $w_1, w_2, ..., w_n$ the following system of quadrics:

$$w_i^2 - w_j^2 = a_j - a_i, \quad i, j = 1, n, \tag{2}$$

where a_i are arbitrary complex numbers. Rank of this system is $n-1$, so substitution:

$$w_i^2 = w - a_i,$$

solves these equations. Moreover if we put $y = \prod_{i=1}^{n} w_i$ we obtain the equation of the hyperelliptic curve $\mathcal{H}$:

$$y^2 = \prod_{i=1}^{n}(w - a_i). \tag{3}$$

Hence equations 2 define embedding of the hyperelliptic curve $\mathcal{H}$ in the linear space C^n. Variable w is a local parameter on the curve and a_i its branching points.

Example. In the $n = 3$ case all these objects have well-known analytical description. Indeed in this case curve under the consideration is elliptic. Its uniformization is made by the Weierstrass $\mathfrak{p}$- function and its derivative: $w = \mathfrak{p}(u)$, $y = 1/2\mathfrak{p}'(u)$. Functions w_i are expressed via Jacobi elliptic functions [10]: $w_1 = \dfrac{1}{sn(u)}$, $w_2 = \dfrac{dn(u)}{sn(u)}$, $w_3 = \dfrac{cn(u)}{sn(u)}$.

Classical Lie algebras. Let $\mathfrak{g}$ denotes one of the classical matrix Lie algebras $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex numbers. For the subsequent we will need special form of their bases. Let us explicitly construct them. Let $I_{i,j} \in Mat(n, C)$ be a matrix defined as follows:

$$(I_{ij})_{ab} = \delta_{ia}\delta_{jb}.$$

Evidently, a basis in the algebra $gl(n)$ could be built from the matrices $X_{ij} \equiv I_{ij}$, $i, j \in 1, ..., n$. The commutation relations in $gl(n)$ will have the following standard form:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j}.$$

The basis in the algebra $so(n)$ could be chosen as: $X_{ij} \equiv I_{ij} - I_{i,j}$, $i, j \in 1, ..., n$, with the following commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \delta_{j,l}X_{k,i} - \delta_{k,i}X_{j,l},$$

and "skew-symmetry" property $X_{ij} = -X_{ji}$.
The basis in the algebra $sp(n)$ consists of the matrices $X_{ij} = I_{ij} - \epsilon_i\epsilon_j I_{-i,-j}$, $i, j \in -n, ..., -1, 1, ..., n$, with the commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \epsilon_i\epsilon_j(\delta_{j,-l}X_{k,-i} - \delta_{k,-i}X_{-j,l}),$$

and additional property $X_{i,j} = -\epsilon_i\epsilon_j X_{-j,-i}$, where $\epsilon_j = \mathrm{sign} j$.

Algebras on the curve. For the basic elements X_{ij} of all three algebras $gl(n)$, $so(n)$ and $sp(n)$ we introduce the following algebra-valued functions on the curve $\mathcal{H}$, or to be more precise on its double covering:

$$X_{ij}^{+}(w) = X_{ij} \otimes w_i w_j, \quad X_{ij}^{-}(w) = X_{ij} \otimes w^{-1} w_i w_j. \tag{4}$$

Here we put $w_{-i} \equiv w_i$ in the case of $sp(n)$. We will need the following definition [11]:
Definition. Infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ is called $\mathbb{Z}$-quasi graded, if there exist such $p, q \in \mathbb{Z}_+$ that:

$$\widetilde{\mathfrak{g}} = \sum_{j\in\mathbb{Z}} \widetilde{\mathfrak{g}}_j, \quad \text{such that} \quad [\widetilde{\mathfrak{g}}_i, \widetilde{\mathfrak{g}}_j] \subset \sum_{k=-p}^{q} \widetilde{\mathfrak{g}}_{i+j+k}. \tag{5}$$

The following theorem holds true:

Theorem 2.1 *(i)Functions $X_{ij}^{+}(w)$ and $X_{ij}^{-}(w)$ generate infinite-dimensional $\mathbb{Z}$ quasi-graded Lie algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$.*
(ii)There exists a decomposition: $\widetilde{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}_{\mathcal{H}}^{+} + \widetilde{\mathfrak{g}}_{\mathcal{H}}^{-}$, where are $\widetilde{\mathfrak{g}}_{\mathcal{H}}^{+}$ and $\widetilde{\mathfrak{g}}_{\mathcal{H}}^{-}$ are generated by $X_{ij}^{+}(w)$ and $X_{ij}^{-}(w)$ correspondingly.

Proof. Let us introduce the following algebra-valued functions on the double covering of the curve $\mathcal{H}$:

$$X_{ij}^{n} = X_{ij} \otimes w^n w_i w_j.$$

To prove the theorem we will need the explicit form of their commutation relations:

Proposition 2.1 *Elements X_{ij}^n form closed algebra with the following commutation relations:*

$$1)\ [X_{ij}^n, X_{ij}^m] = \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} \quad \textit{for the } gl(n) \tag{6}$$

$$\begin{aligned} 2)\ [X_{ij}^n, X_{ij}^m] &= \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + \delta_{jl} X_{ki}^{n+m+1} - \delta_{ik} X_{jl}^{n+m+1} \\ &+ a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} + a_i \delta_{ik} X_{jl}^{n+m} - a_j \delta_{jl} X_{ki}^{n+m} \quad \textit{for the } so(n) \end{aligned} \tag{7}$$

$$\begin{aligned} 3)\ [X_{ij}^n, X_{ij}^m] &= \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + \epsilon_i \epsilon_j (\delta_{j-l} X_{k-i}^{n+m+1} - \delta_{i-k} X_{j-l}^{n+m+1}) \\ + a_i \delta_{il} X_{kj}^{n+m} &- a_j \delta_{kj} X_{il}^{n+m} + a_i \epsilon_i \epsilon_j (a_i \delta_{i-k} X_{j-l}^{n+m} - a_j \delta_{j-l} X_{k-i}^{n+m}) \quad \textit{for the } sp(n). \end{aligned} \tag{8}$$

Let us put

$$\widetilde{\mathfrak{g}}_{\mathcal{H}} = \mathrm{Span}_{\mathbb{C}}\{X_{ij}^m, m \in \mathbb{Z}\}.$$

From the Proposition 2.1 follows that $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ is quasi-graded. Besides, let us define the following subspaces:

$$\widetilde{\mathfrak{g}}^{+}_{\mathcal{H}} = \mathrm{Span}_{\mathbb{C}}\{X^{m}_{ij}, m \in \mathbb{Z}_{+} \cup 0\}, \quad \widetilde{\mathfrak{g}}^{-}_{\mathcal{H}} = \mathrm{Span}_{\mathbb{C}}\{X^{-m}_{ij}, m \in \mathbb{Z}_{+}\}.$$

From the Proposition 2.1 follows, that they are subalgebras. Taking into account that $X^{+}_{ij}(w) = X^{0}_{ij}$, $X^{-}_{ij}(w) = X^{-1}_{ij}$ and the Proposition 2.1, it is easy to see that elements $X^{+}_{ij}(w)$ generate subalgebra $\widetilde{\mathfrak{g}}^{+}_{\mathcal{H}}$ and elements $X^{-}_{ij}(w)$ generate subalgebra $\widetilde{\mathfrak{g}}^{-}_{\mathcal{H}}$. Hence $X^{+}_{ij}(w)$ and $X^{-}_{ij}(w)$ generate $\widetilde{\mathfrak{g}}_{\mathcal{H}}$. Theorem is proved.

Example. Let $\mathfrak{g} = so(3)$. In this case constructed algebra will coincide with the "even" subalgebra of the algebra of hidden symmetry of Landau- Lifschits equations [9]. Indeed, putting $X_k \equiv \epsilon_{ijk} X_{ij}$, we obtain the following commutation relations:

$$[X^{n}_{i}, X^{m}_{j}] = \epsilon_{ijk} X^{n+m+1}_{k} + \epsilon_{ijk} a_k X^{n+m}_{k}.$$

Remark. From the Proposition 2.1 follows, that in the rational degeneration of $\mathcal{H}$:

$$y^2 = w^n,$$

i.e when $a_i = 0$, we obtain, that $w_i = \sqrt{w}$ and, hence, $\widetilde{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}$, where $\widetilde{\mathfrak{g}}$ is ordinary loop algebra.

2.2 Central extension

To have the full analogy with the loop algebra, and keeping in mind possible application to quantum integrable systems, in this subsection we define central extensions of the algebras $\widetilde{\mathfrak{g}}_{\mathcal{H}}$. First we remind the following definition [11]:
Definition. Cocycle χ on the quasigraded algebra is called local if

$$\chi(X^{m}_{ij}, X^{n}_{kl}) = 0, \quad \text{for all} \quad |n+m| > K \quad \text{and for some} \quad K \in \mathbb{Z}_{+}.$$

Let $(X|Y) = c_n tr XY$, where $c_n = 2n$ for $gl(n)$, $c_n = (n-2)$ for $so(n)$, $c_n = (2n+2)$ for $sp(n)$, be a standard invariant form on the classical Lie algebras $gl(n)$, $so(n)$ or $sp(n)$. Let us define on $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ following bilinear form:

$$\chi(X(w), Y(w) = \oint_{\gamma} (X(w)|\frac{dY(w)}{dw})dw, \tag{9}$$

where 1-cycle γ in the complex plane of variable w encircles point $w = 0$.

Although the above bilinear form is standard in the theory of loop algebras, it is not straightforward fact that it determines the cocycle on $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ due to the fact, that w_i are not one-valued functions of w. We prove this in the following theorem:

Theorem 2.2 *(i)Bilinear form χ is skew-symmetric and satisfies the properties of cocycle, and hence, determines a central extension of $\widetilde{\mathfrak{g}}_{\mathcal{H}}$: $\widehat{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}_{\mathcal{H}} + \mathbb{C}c$ by the following formula:*

$$[X(w) + \alpha c, Y(w) + \beta c]' = [X(w), Y(w)] + \chi(X(w), Y(w))c.$$

(ii) Cocycle χ is local and its values on the bases elements are calculated by the following formulas:

$$\chi(X_{ij}^m, X_{kl}^n) = \chi^{mn}(a_i, a_j)(X_{ij}|X_{kl}), \quad \text{where} \tag{10}$$

$$\chi^{mn}(a_i, a_j) = (n+1)\delta_{n+m+2,0} - (n+1/2)(a_i + a_j)\delta_{n+m+1,0} + n a_i a_j \delta_{n+m,0}. \tag{11}$$

Example. In the case of rational degeneration $a_i = 0$ we obtain the following cocycle on the loop algebra:

$$\chi(X_{ij}^m, X_{kl}^n) = (n+1)\delta_{n+m+2,0}(X_{ij}|X_{kl}).$$

It pass to the standard one after renaming the indices: $m+1 \to m$, $n+1 \to n$.

2.3 Coadjoint representation

To define the coadjoint representation we have to define $\widetilde{\mathfrak{g}}_{\mathcal{H}}^*$. We assume, that $\widetilde{\mathfrak{g}}_{\mathcal{H}}^* \subset \mathfrak{g} \otimes A$, where A is an algebra of function on the double covering of the curve $\mathcal{H}$. Let us define invariant pairing between $L(w) \in \widetilde{\mathfrak{g}}_{\mathcal{H}}^*$ and $X(w) \in \widetilde{\mathfrak{g}}_{\mathcal{H}}$ in the following way:

$$\langle X(w), L(w) \rangle_f = c_n res_{w=0} f^{-1}(w) y^{-1}(w)(X(w)|L(w)), \tag{12}$$

where $f(w)$ is arbitrary function on the curve $\mathcal{H}$. It is easy to show, that element dual to X_{ij}^{-m} with respect to this pairing is $Y_{ij}^m \equiv (X_{ij}^{-m})^* = \dfrac{w^{m-1} f(w) y(w)}{w_i w_j} X_{ij}^*$. Hence the general element of the dual space has the following form:

$$L(w) = \sum_{m \in Z} \sum_{i,j=1}^{n} l_{ij}^m \frac{w^{m-1} f(w) y(w)}{w_i w_j} X_{ij}^* \tag{13}$$

Coadjoint action of algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ on its dual space $\widetilde{\mathfrak{g}}_{\mathcal{H}}^*$ coincides with commutator:

$$ad_{X(w)}^* L(w) = [L(w), X(w)]. \tag{14}$$

Explicitly coadjoint action of algebra has the following form:

1) $[X_{ij}^n, Y_{kl}^m] = \delta_{kj} Y_{il}^{n+m+1} - \delta_{il} Y_{kj}^{n+m+1} + a_j \delta_{il} Y_{kj}^{n+m} - a_i \delta_{kj} Y_{il}^{n+m}$ for the $gl(n)$

2) $[X^n_{ij}, Y^m_{kl}] = \delta_{kj} Y^{n+m+1}_{il} - \delta_{il} Y^{n+m+1}_{kj} + \delta_{jl} Y^{n+m+1}_{ki} - \delta_{ik} Y^{n+m+1}_{jl}$
$+ a_j \delta_{il} Y^{n+m}_{kj} - a_i \delta_{kj} Y^{n+m}_{il} + a_j \delta_{ik} Y^{n+m}_{jl} - a_i \delta_{jl} Y^{n+m}_{ki}$ for the $so(n)$

3) $[X^n_{ij}, Y^m_{kl}] = \delta_{kj} Y^{n+m+1}_{il} - \delta_{il} Y^{n+m+1}_{kj} + \epsilon_i \epsilon_j (\delta_{j-l} Y^{n+m+1}_{k-i} - \delta_{i-k} Y^{n+m+1}_{j-l})$
$+ a_j \delta_{il} Y^{n+m}_{kj} - a_i \delta_{kj} Y^{n+m}_{il} + \epsilon_i \epsilon_j (a_j \delta_{i-k} Y^{n+m}_{j-l} - a_i \delta_{j-l} Y^{n+m}_{k-i})$ for the $sp(n)$.

It is evident from the above formulas, that $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ is a quasi-graded $\widetilde{\mathfrak{g}}_{\mathcal{H}}$-module:

$$\widetilde{\mathfrak{g}}^*_{\mathcal{H}} = \sum_{m \in \mathbb{Z}} (\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_m,$$

where $(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_m = Span_C\{Y^m_{ij} | i, j = 1, ..., n\}$.
Remark 1. Putting $f(w) = y(w)$ in the definition of the elements of dual space we obtain, that $\widetilde{\mathfrak{g}}^*_{\mathcal{H}} \subseteq \widetilde{\mathfrak{g}}_{\mathcal{H}}$. If $f(w) \neq y(w)$, then generally speaking, spaces $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ and $\widetilde{\mathfrak{g}}_{\mathcal{H}}$, viewing as subspaces in the algebra of functions on the double covering of the curve $\mathcal{H}$, do not coincide.
Remark 2. If $f = 1$ and $\mathfrak{g} = so(n)$ elements of $\widetilde{\mathfrak{g}}_{\mathcal{H}} + \widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ form a closed algebra.This will be analogue of the algebra of hidden symmetry of Landau- Lifschits equations [7]. Unfortunately, for $n > 4$ it does not admit a Kostant-Adler scheme and can not be used for the construction of integrable systems. That is why we will not consider it here.
From the explicit form of coadjoint action (14) follows the next statement:

Proposition 2.2 *Functions* $I^k_m(L(w)) = res_{w=0} w^{-m-1} Tr L(w)^k$*, where* $m \in \mathbb{Z}$*, are invariants of coadjoint representation.*

Hence constructed Lie algebras not only admit decomposition into the direct sum of two subalgebras but also possess infinite number of invariant functions. This permits to use them in construction of integrable systems.

3 Integrable systems from hyperelliptic algebras

3.1 Poisson structures and Poisson subspaces.

First Lie-Poisson structure. In the space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ it is possible to define many Lie-Poisson structures using different pairings. We will use the pairing (12) with $f(w) = w$:

$$\langle X(w), L(w) \rangle_{-1} = c_n res_{w=0} w^{-1} y^{-1}(w) (X(w) | L(w)). \qquad (15)$$

It defines brackets on $P(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})$ in the following way:

$$\{F(L), G(L)\} = \sum_{l,m \in Z} \sum_{i,j,p,s=1}^{n} \langle L(w), [X^{-l}_{ij}, X^{-m}_{ps}] \rangle_{-1} \frac{\partial G}{\partial l^l_{ij}} \frac{\partial F}{\partial l^m_{ps}} \qquad (16)$$

The following Corollary of the Proposition 2.2 holds true:

Proposition 3.1 *Functions $I^k_m(L(w))$ are central for brackets $\{\ ,\ \}$.*

Taking into account that $l^m_{ij} = \langle L(w), X^{-m}_{ij}\rangle_{-1}$, it is easy to show, that for the coordinate functions l^m_{ij} these brackets will have the following form:

$$1)\ \{l^n_{ij}, l^m_{ij}\} = \delta_{kj}l^{n+m-1}_{il} - \delta_{il}l^{n+m-1}_{kj} + a_i\delta_{il}l^{n+m}_{kj} - a_j\delta_{kj}l^{n+m}_{il} \quad \text{for the } gl(n) \tag{17}$$

$$\begin{aligned} 2)\ \{l^n_{ij}, l^m_{ij}\} = {} & \delta_{kj}l^{n+m-1}_{il} - \delta_{il}l^{n+m-1}_{kj} + \delta_{jl}l^{n+m-1}_{ki} - \delta_{ik}l^{n+m-1}_{jl} \\ & + a_i\delta_{il}l^{n+m}_{kj} - a_j\delta_{kj}l^{n+m}_{il} + a_i\delta_{ik}l^{n+m}_{jl} - a_j\delta_{jl}l^{n+m}_{ki} \quad \text{for the } so(n) \end{aligned} \tag{18}$$

$$\begin{aligned} 3)\ \{l^n_{ij}, l^m_{ij}\} = {} & \delta_{kj}l^{n+m-1}_{il} - \delta_{il}l^{n+m-1}_{kj} + \epsilon_i\epsilon_j(\delta_{j-l}l^{n+m-1}_{k-i} - \delta_{i-k}l^{n+m-1}_{j-l}) \\ & + a_i\delta_{il}l^{n+m}_{kj} - a_j\delta_{kj}l^{n+m}_{il} + \epsilon_i\epsilon_j(a_i\delta_{i-k}l^{n+m}_{j-l} - a_j\delta_{j-l}l^{n+m}_{k-i}) \quad \text{for the } sp(n). \end{aligned} \tag{19}$$

From the explicit form of the Poisson brackets follows, that Lie-Poisson brackets in the subspaces $(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_- = \sum\limits_{m=0}^{\infty}(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_{-m}$ and $(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_+ = \sum\limits_{m=1}^{\infty}(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_m$, up to the reverse of the sign of the upper indices, repeat commutation relations of the algebras $\widetilde{\mathfrak{g}}^+_{\mathcal{H}}$ and $\widetilde{\mathfrak{g}}^-_{\mathcal{H}}$ correspondingly. This, of course, is the result of the following duality: $(\widetilde{\mathfrak{g}}^+_{\mathcal{H}})^* = (\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_-$, $(\widetilde{\mathfrak{g}}^-_{\mathcal{H}})^* = (\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_+$.

Second Lie-Poisson structure. Let us introduce into the space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ a new Poisson brackets $\{\ ,\ \}_0$, which are a Lie-Poisson brackets for the algebra $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$, where $\widetilde{\mathfrak{g}}^0_{\mathcal{H}} = \widetilde{\mathfrak{g}}^-_{\mathcal{H}} \ominus \widetilde{\mathfrak{g}}^+_{\mathcal{H}}$. Explicitly, this brackets have the following form:

$$\{l^n_{ij}, l^m_{kl}\}_0 = -\{l^n_{ij}, l^m_{kl}\},\ n,m > 0, \quad \{l^n_{ij}, l^m_{kl}\}_0 = \{l^n_{ij}, l^m_{kl}\},\ n,m \leq 0,$$
$$\{l^n_{ij}, l^m_{kl}\}_0 = 0, m \leq 0, n > 0 \text{ or } n \leq 0, m > 0\ .$$

Let subspace $\mathcal{M}_{s,p} \subset \widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ be defined as follows:

$$\mathcal{M}_{s,p} = \sum_{m=-s+1}^{p}(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_m.$$

Brackets $\{\ ,\ \}_0$ could be correctly restricted to $\mathcal{M}_{s,p}$. It follows from the next Proposition:

Proposition 3.2 *Subspaces $\mathcal{J}_{p,s} = \sum\limits_{m=-\infty}^{-p-1}(\widetilde{\mathfrak{g}}_{\mathcal{H}})_m + \sum\limits_{m=s}^{\infty}(\widetilde{\mathfrak{g}}_{\mathcal{H}})_m$ are ideals in $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$.*

Proof. It follows from the explicit form of commutation relations in $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$.

Now we are ready to prove the following important theorem:

Theorem 3.1 *Functions $\{I_m^k(L)\}$ commutes with respect to the restriction of the brackets $\{\ ,\ \}_0$ on $\mathcal{M}_{s,p}$.*

Proof. It follows from the combination of Kostant-Adler scheme and previous Proposition. Indeed, due to the fact, that $\{I_m^k(L)\}$ are Casimir functions on $\widetilde{\mathfrak{g}}_{\mathcal{H}}^*$ they form a commutative subalgebra with respect to the brackets $\{\ ,\ \}_0$ [4]. Hence they will stay commutative after the restriction to $\mathcal{M}_{s,p} = (\widetilde{\mathfrak{g}}_{\mathcal{H}}^0/\mathcal{J}_{p,s})^*$, due to the fact, that projection onto quotient algebra is a canonical homomorphism.

Hamiltonian equations will have the following form:

$$\frac{dl_{ij}^k}{dt} = \{l_{ij}^k, H(l_{kl}^m)\}_0, \tag{20}$$

where hamiltonian H is one of the functions I_m^k or their linear combination. These equations could be written in the Lax form [6]:

$$\frac{dL(w)}{dt} = [L(w), M(w)], \tag{21}$$

where $L(w) \in \mathcal{M}_{s,p}$, and second operator is defined as follows: $M(w) = (P_- - P_+)\nabla H(L(w))$. Here $P_\pm$ are projection operators on the subalgebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}^\pm$,

$$\nabla H(L(w)) = \sum_{k=-p}^{s-1} \sum_{ij=1}^{n} \frac{\partial H}{\partial l_{ij}^k} X_{ij}^{-k} \tag{22}$$

is an algebra-valued gradient of H.

4 Integrable systems in finite-dimensional quotients.

The most interesting from the physical point of view examples usually arise in the spaces $\mathcal{M}_{s,p}$ with small s and p. We will consider the case $|s+p| \le 2$. We will assume, that curve $\mathcal{H}$ is nondegenerated, i.e. $a_i \ne a_j$ for $i \ne j$. This requirement is necessary for completeness of the family of constructed commuting functions.

The basic algebra in all examples will be $\mathfrak{g} = so(n)$, but analogous results are valid for $\mathfrak{g} = gl(n)$ and $\mathfrak{g} = sp(n)$. We chose for all examples $so(n)$ algebra only because correspondent integrable systems are the most direct generalizations of classical integrable systems connected with $so(4)$ and $e(3)$.

4.0.1 Generalized interacting tops.

Let us consider subspace $\mathcal{M}_{1,1}$. In the case $a_i \ne 0$, and as it follows from the arguments below, $(\widetilde{\mathfrak{g}}_{\mathcal{H}}^+/\mathcal{J}_{1,1}) \simeq \mathfrak{g} \oplus \mathfrak{g}$, hence $\mathcal{M}_{1,1} = (\mathfrak{g} \oplus \mathfrak{g})^*$. Corresponding Lax

operator $L(w) \in \mathcal{M}_{0,1}$ has the following form:

$$L(w) = \sum_{i,j=1}^{n} (l_{ij}^{(0)} + w l_{ij}^{(1)}) \frac{y(w)}{w_i w_j} X_{ij}^*.$$

Let us again consider $so(n)$ case and put $X_{ij}^* = X_{ij}$. Lie-Poisson brackets between the coordinate functions $l_{ij}^{(1)}$ have the following form:

$$\{l_{ij}^{(0)}, l_{kl}^{(0)}\} = -a_i \delta_{il} l_{kj}^{(0)} + a_j \delta_{kj} l_{il}^{(0)} - a_i \delta_{ik} l_{jl}^{(0)} + a_j \delta_{jl} l_{ki}^{(0)},$$

$$\{l_{i,j}^{(1)}, l_{k,l}^{(1)}\} = \delta_{k,j} l_{i,l}^{(1)} - \delta_{i,l} l_{k,j}^{(1)} + \delta_{j,l} l_{k,i}^{(1)} - \delta_{k,i} l_{j,l}^{(1)},$$

$$\{l_{ij}^{(0)}, l_{kl}^{(1)}\} = 0.$$

Putting $b_i = a_i^{1/2}$ and making the change of variables: $l_{ij} = l_{i,j}^{(1)}$, $m_{ij} = \dfrac{l_{ij}^{(0)}}{b_i b_j}$, we obtain canonical coordinates of the direct sum of two algebras $so(n)$:

$$\{m_{i,j}, m_{k,l}\} = \delta_{k,j} m_{i,l} - \delta_{i,l} m_{k,j} + \delta_{j,l} m_{k,i} - \delta_{k,i} m_{j,l},$$

$$\{l_{i,j}, l_{k,l}\} = \delta_{k,j} l_{i,l} - \delta_{i,l} l_{k,j} + \delta_{j,l} l_{k,i} - \delta_{k,i} l_{j,l},$$

$$\{l_{ij}, m_{kl}\} = 0.$$

Commuting integrals are constructed using expansions in the powers of w of the functions: $I_k(w) = Tr(L(w))^k$. As in the previous example, we are mainly interested in the quadratic integrals. Let

$$h(w) \equiv I_2(w) = \sum_{s=0}^{n} h_s(l_{ij}^{(1)}) w^s = \sum_{ij} (\prod_{k \neq i,j} (w - a_k))(l_{ij}^{(0)} + w l_{ij}^{(1)})^2.$$

By straightforward calculations we obtain:

$$h_0 = (-1)^{n-2} \sum_{i,j=1}^{n} \frac{a_1 a_2 \ldots\ a_n}{a_i a_j} (l_{ij}^{(0)})^2$$

$$h_1 = (-1)^{n-1} \sum_{i,j=1}^{n} (\sum_{k \neq i,j} \frac{a_1 a_2 \ldots\ a_n}{a_k}) \frac{(l_{ij}^{(0)})^2}{a_i a_j} - 2 \frac{a_1 a_2 \ldots\ a_n}{a_i a_j} l_{ij}^{(0)} l_{ij}^{(1)}$$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$$h_{n-1} = - \sum_{i,j=1}^{n} (\sum_{k=1}^{n} a_k - (a_i + a_j))(l_{ij}^{(1)})^2 - 2 l_{ij}^{(0)} l_{ij}^{(1)}$$

$$h_n = \sum_{i,j=1}^{n} (l_{ij}^{(1)})^2.$$

Making the described above replacement of variables we will have:

$$h_0 = (-1)^{n-2}(b_1^2 b_2^2 ... \ b_n^2) \sum_{i,j=1}^{n} m_{ij}^2$$
$$h_1 = (-1)^{n-1} \sum_{i,j=1}^{n} (\sum_{k \neq i,j} \frac{b_1^2 b_2^2 ... \ b_n^2}{b_k^2})(m_{ij})^2 - 2\frac{b_1^2 b_2^2 ... \ b_n^2}{b_i b_j} m_{ij} l_{ij}$$
$$\dots\dots\dots\dots\dots\dots$$
$$h_{n-1} = -\sum_{i,j=1}^{n} (\sum_{k=1}^{n} b_k^2 - (b_i^2 + b_j^2)) l_{ij}^2 - 2 b_i b_j m_{ij} l_{ij}$$
$$h_n = \sum_{i,j=1}^{n} (l_{ij})^2.$$

It is evident, that functions h_0 and h_n are invariants. Functions h_1,...,h_{n-1} generate non-trivial hamiltonian flows. For the hamiltonian of the generalized interacting rigid bodies we can take either h_{n-1} or h_1. Correspondent M-operator and Lax equations are calculated straightforwardly.

Steklov-Veselov ($n = 3$) case. In this case, making standard of variables $l_i = \epsilon_{ijk} l_{ij}$, $m_i = \epsilon_{ijk} m_{ij}$ we obtain the following set of commuting functions:

$$h_0 = \sum_{k=1}^{n} m_k^2, \quad h_3 = \sum_{k=1}^{n} l_k^2,$$
$$h_1 = (\sum_{k=1}^{n} (\frac{b_1 b_2 b_3}{b_k^2})(m_k)^2 - 2 b_k m_k l_k),$$
$$h_2 = \sum_{i,j=1}^{n} (b_k^2 l_k^2 - 2\frac{b_1 b_2 b_3}{b_k} m_k l_k).$$

Here h_0, h_3 — invariant functions and h_1,h_2 are two independent integrals discovered by Veselov [13]. Commutation relations between coordinates l_k and m_k of $so(4)$ are standard:

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \ \ \{m_i, m_j\} = \epsilon_{ijk} m_k, \ \ \{m_i, l_j\} = 0.$$

4.0.2 Generalized Steklov-Ljapunov case.

The last class of integrable systems we wish to consider here will be integrable systems that generalize classical integrable system of Steklov-Ljapunov on $e(3) = so(3) + R^3$. Let us consider subspace $\mathcal{M}_{0,2} = (\widetilde{\mathfrak{g}}^{+}_{\mathcal{H}}/\mathcal{J}_{2,0})^*$. It is easy to show that $\mathcal{M}_{0,2} = (\mathfrak{g} + \mathfrak{g})^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{0,2}$ has the following form:

$$L(w) = w(\sum_{i,j=1}^{n} (l_{ij}^{(1)} + w l_{ij}^{(2)}) \frac{y(w)}{w_i w_j} X_{ij}^*).$$

We will again be concentrated on $\mathfrak{g} = so(n)$ case and put $X_{ij}^* = X_{ij}$. Lie-Poisson brackets between coordinate functions are following:

$$\{l_{ij}^{(1)}, l_{ij}^{(1)}\} = \delta_{kj} l_{il}^{(1)} - \delta_{il} l_{kj}^{(1)} + \delta_{jl} l_{ki}^{(1)} - \delta_{ik} l_{jl}^{(1)} + a_i \delta_{il} l_{kj}^{(2)} - a_j \delta_{kj} l_{il}^{(2)} + a_i \delta_{ik} l_{jl}^{(2)} - a_j \delta_{jl} l_{ki}^{(2)},$$

$$\{l_{ij}^{(1)}, l_{ij}^{(2)}\} = \delta_{kj} l_{il}^{(2)} - \delta_{il} l_{kj}^{(2)} + \delta_{jl} l_{ki}^{(2)} - \delta_{ik} l_{jl}^{(2)}, \quad \{l_{ij}^{(2)}, l_{ij}^{(2)}\} = 0.$$

Commuting integrals are constructed using expansion in the powers of w of the functions: $I_k(w) = Tr(L(w))^k$. We are again interested mainly in quadratic integrals. Let

$$h(w) \equiv I_2(w) = w^2 \sum_{s=0}^{n} h_{s+2}(l_{ij}^{(1)}) w^s = w^2 \sum_{ij} (\prod_{k \neq i,j} (w - a_k))(l_{ij}^{(1)} + w l_{ij}^{(2)})^2.$$

Direct calculations give:

$$h_2 = (-1)^{n-2} \sum_{i,j=1}^{n} \frac{a_1 a_2 \ldots\ a_n}{a_i a_j} (l_{ij}^{(1)})^2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h_{n+1} = (-1)(\sum_{i,j=1}^{n} (\sum_{k=1}^{n} a_k - (a_i + a_j))(l_{ij}^{(2)})^2 - 2 l_{ij}^{(1)} l_{ij}^{(2)})$$

$$h_{n+2} = \sum_{i,j=1}^{n} (l_{ij}^{(2)})^2.$$

Change of variables: $l_{ij}^{(1)} = l_{ij} - 1/2(a_i + a_j)p_{ij}$, $l_{ij}^{(2)} = p_{ij}$ transform described above brackets to the canonical brackets on the half-direct sum $so(n) + so(n)$:

$$\{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj} + \delta_{jl} l_{ki} - \delta_{ik} l_{jl},$$

$$\{l_{ij}, p_{kl}\} = \delta_{kj} p_{il} - \delta_{il} p_{kj} + \delta_{jl} p_{ki} - \delta_{ik} p_{jl},$$

$$\{p_{ij}, p_{kl}\} = 0.$$

After such transformation we obtain the following set of hamiltonians:

$$h_2 = (-1)^{n-2} \sum_{i,j=1}^{n} \frac{a_1 a_2 \ldots\ a_n}{a_i a_j} (l_{ij} - 1/2(a_i + a_j)p_{ij})^2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h_{n+1} = (-1)(\sum_{k=1}^{n} a_k)(\sum_{i,j=1}^{n} p_{ij}^2) - 2(\sum_{i,j=1}^{n} l_{ij} p_{ij})$$

$$h_{n+2} = \sum_{i,j=1}^{n} p_{ij}^2.$$

Last two functions are invariant functions. First $n-1$ give non-trivial flows on $\mathfrak{g}^*$. We will chose function $H = h_2$ for the hamiltonian function. Correspondent M operator is:

$$M(w) = 2 \sum_{i,j=1}^{n} \frac{a_1 a_2 \ldots\ a_n}{a_i a_j} (l_{ij} - 1/2(a_i + a_j)p_{ij}) w^{-1} w_i w_j X_{ij}.$$

Lax equation have the standard form (1).

Steklov-Ljapunov ($n=3$) case. In this case, making standard replacement of variables $l_k = \epsilon_{ijk} l_{ij}$, $p_k = \epsilon_{ijk} p_{ij}$ we obtain:

$$H = (-1)^{n-2} \sum_{k=1}^{n} a_k (l_k - 1/2(a_1 + a_2 + a_3 - a_k) p_k)^2.$$

Up to the rescaling of momenta p_i: $p_i \to 2\sigma p_i$ hamiltonian H coincides with the hamiltonian of Steklov-Ljapunov system in the form of Kotter [15].

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